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Matrix Transformations on Cesaro Vector-Valued Sequence Space

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Abstract. The purpose of this paper is to find β-dual of Cesaro vector-valued sequence space and give matrix characterizations from $Ces(X,p)$ into sequence spaces $\ell(q)$, $\ell_\infty(q)$, $M_\infty(q)$ and $\underline{\ell_\infty}(q)$ where $p=(p_k)$ is a bounded sequence of positive real numbers such that $p_k>1$ for all $k\in N$.

1. Introduction

Let $(X,\|.\|)$ be a Banach space with a scalar field K, the space of all sequences in X is denote by $W(X)$ and let $\Phi(X)$ denote the space of all finite sequences in X. When $X=\mathbb{R}$ or $\mathcal{C}$, the corresponding spaces are written as W and Φ. Let N be the set of all natural numbers, we write $x=(x_k)$ with $x_k\in X$ for all $k\in N$. A sequence space in X is linear subspace of $W(X)$. Let $p=(p_k)$ be a bounded sequence of positive real numbers, the X-valued sequence space $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $Ces(X,p)$, $\underline{\ell_\infty}(X,p)$, $E_r(X,p)$ and $F_r(X,p)$ are define by:

$$c_0(X,p)=\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\},$$
$$c(X,p)=\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0 \text{ for some } a\in X\},$$
$$\ell_\infty(X,p)=\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\},$$

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$$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\},$$

$$Ces(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} (\frac{1}{k}\sum_{n=1}^{k} \|x_n\|)^{p_k} < \infty\},$$

$$\underline{\ell_\infty}(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|\delta_k x_k\| = 0 \quad \text{for each}(\delta_k) \in c_0\},$$

$$E_r(X,p) = \{x = (x_k) : \sup_k k^{-r}\|x_k\|^{p_k} < \infty\}, \text{ and}$$

$$F_r(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} k^r\|x_k\|^{p_k} < \infty\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p), Ces(p), \underline{\ell_\infty}(p), E_r(p)$ and $F_r(p)$ respectively.

Grosse and Erdmann [2-3] investigated and gave characterization for infinite matrices to transform between sequence spaces of Maddox. In 1993, F. M. Khan and M. A. Khan[4] gave characterization of infinite matrices of Cesàro sequence space ($Ces(p,s)$) into the space of convergent series (cs) and the space of bounded series (bs). Wu and Liu [9] gave the matrix transformations from X-valued sequence spaces $c_0(X,p), \ell_\infty(X,p)$ and $\ell(X,p)$ into scalar-valued sequence spaces $c_0(q)$ and $\ell_\infty(q)$. S. Suantai[6, 7, 8] gave characterization of infinite matrices mapping Nakano vector-valued sequence space $\ell(X,p)$ into any BK-space, ℓ_∞ and $\ell_\infty(q)$. In [1] C. Sudsukh characterized an infinite matrix that transform Maddox vector-valued sequence space into Nakano sequence space and Nakano vector-valued sequence space into Maddox sequence space. In [3] S. Kongnual characterized the matrix transformation of bounded variation vector-valued sequence space into Maddox sequence space.

However, there are many open problems about matrix transformations from vector-valued sequence spaces into scalar-valued sequence spaces. In this paper we study matrix transformations of Cesaro vector-valued sequence space $Ces(X,p)$ into sequence spaces $\ell(q)$, $\ell_\infty(q)$, $M_\infty(q)$ and $\underline{\ell_\infty}(q)$.

2. Definitions and lemmas

For $z \in X$ and $k \in N$, we let $e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined by

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a **K-space** if for each $n \in N$ the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E,τ) is a Fréchet(Banach, LF-, LB-) space, then E is called an $FK-(BK-, LFK-, LBK-)$ space. Now, suppose that E contains $\Phi(X)$. Then E is said to have **property AB** if the set $\{\Sigma_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It said to have **property AK** if $\Sigma_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x = (x_k) \in E$. It has **property AD** if $\Phi(X)$ is dense in E. Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to **map** E **into** F, written $A : E \to F$ if for each $x = (x_k) \in E, A_n(x) = \Sigma_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and if the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E, F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E, F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u_k})$.

In C. Sudsukh [1], this lemma is useful to characterize condition of matrix transformations.

Lemma 2.1. *Let $E \subseteq W(X)$ be an FK-space with AK property and F an FK-space of scalar sequences. Then, for an infinite matrix $A = (f_k^n), A : E \to F$ if and only if*

(1) *for each $n \in N, \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,*

(2) *for each $k \in N, (f_k^n(z))_{n=1}^{\infty} \in F$ for all $z \in X$, and*

(3) *$A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E.*

3. Some auxiliary results

In this part we first give useful results that concern with $\beta - dual$ of $Ces(X, p)$.

Proposition 3.1. *Let (f_k) be a sequence of continuous linear functional on X and $p = (p_k)$ of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in Ces(X, p)$ if and only if $\sum_{k=1}^{\infty} (\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty$ for some $B \in N$.*

Proof. Suppose that $\sum_{k=1}^{\infty} (\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty$ for some $B \in N$ then for each $x = (x_k) \in Ces(X, p)$

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\leq \sum_{k=1}^{\infty} \|f_k\|.k.B^{-1}B.\frac{1}{k}\|x_k\| \\
&\leq \sum_{k=1}^{\infty} [(\|f_k\|.k)^{t_k} B^{-t_k} + B^{p_k}(\frac{1}{k}\|x_k\|)^{p_k}] \\
&\leq \sum_{k=1}^{\infty} [(\sup_k \|f_k\|.k)^{t_k} B^{-t_k} + B^{p_k}(\frac{1}{k}\|x_k\|)^{p_k}] \\
&= \sum_{k=1}^{\infty} (\sup_n \|f_n\|.k)^{t_k} B^{-t_k} + \sum_{k=1}^{\infty} B^{p_k}(\frac{1}{k}\|x_k\|)^{p_k}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} + B^{\sup_k p_k} \sum_{k=1}^{\infty}(\frac{1}{k}\|x_k\|)^{p_k} \\
&\leq \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} + B^{G} \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{k=1}^{k}\|x_k\|)^{p_k} \\
&< \infty.
\end{aligned}$$

Thus $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for each $x = (x_k) \in Ces(X, p)$. Choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|, \forall k \in N$. Since $(t_k x_k) \in Ces(X, p)$ by assumption we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges then

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \quad \text{for all} \quad x = (x_k) \in Ces(X, p). \tag{3.1}$$

We want to show that $\exists B \in N$ such that $\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty$. On contrary, suppose that

$$\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} b^{-t_k} = \infty, \quad \forall b \in N. \tag{3.2}$$

By (3.2) implies that for each $k_0 \in N$

$$\sum_{k>k_0}(\sup_n \|f_n\|.k)^{t_k} {b_1}^{-t_k} = \infty, \quad \forall b_1 \in N. \tag{3.3}$$

From (3.3) we can choose $b_2 > b_1$ and $b_2 > 2^2$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\leq k_2}(\sup_n \|f_n\|.k)^{t_k} {b_2}^{-t_k} > k^2. \tag{3.4}$$

Doing in this way go on, we have sequence $1 = k_0 < k_1 < k_2 < \dots$ and $b_1 < b_2 < ,\dots, b_i > 2^i$ such that

$$\sum_{k_{i-1}<k\leq k_i}(\sup_n \|f_n\|.k)^{t_k} {b_i}^{-t_k} > k^2.$$

Choose x_k in X with $\|x_k\| = 1$ such that $\sum_{k_{i-1}<k\leq k_i}(\sup_n |f_n(x_n)|.k)^{t_k} {b_i}^{-t_k} > k^2, \forall i \in N$. Let $a_i = \sum_{k_{i-1}<k\leq k_i}(\sup_n |f_n(x_n)|.k)^{t_k} {b_i}^{-t_k}$ and $y = (y_k)$, $y_k = a_i^{-1}(\sup_n |f_n(x_n)|.k)^{t_k} |f_k(x_k)|^{-1} x_k$. Then $y \in Ces(X, p)$. Let $\alpha = (\sup_n |f_n(x_n)|)$ and $G = \sup_k p_k$, we can separate two cases.

Case $\alpha < 1$;

$$[\sum_{k_{i-1}<k\leq k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k}]^{\frac{1}{G}} = [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}\|a_i^{-1}\alpha^{t_j}j^{t_j}|f_j(x_j)|^{-1}.x_j\|)^{p_k}]^{\frac{1}{G}}$$

$$= [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}\alpha^{t_j}j^{t_j}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ;\|x_j\| = 1$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}1.j^{t_j}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ;\alpha^{t_j} \leq \alpha < 1,\ t_j > 1$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}1.k^{\sup_k t_k}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ; j^{t_j} \leq k^{t_j} \leq k^{\sup_k t_k}, \forall k \in N.$$

Let $K_{1,k}$ and $K_{2,k}$ are partitions of $\{1, 2, ..., k\}$, if $j \in K_{1,k}$ then $|f_j(x_j)| < 1$ and if $j \in K_{2,k}$ then $|f_j(x_j)| \geq 1$ for all $j = 1, 2, ..., k$. So we have

$$= [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}}a_i^{-1}k^L|f_j(x_j)|^{-1} + \sum_{j\in K_{2,k}}a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}; L = \sup_k t_k$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}}a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$+[\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}}a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}}a_i^{-1}k^L C)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}}a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$;|f_j(x_j)|^{-1} \leq \max_j |f_j(x_j)|^{-1} = C,\ \text{where } C > 1$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}}a_i^{-1}k^L C)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}}a_i^{-1}k^L.1)^{p_k}]^{\frac{1}{G}}$$

$$;|f_j(x_j)|^{-1} \leq 1$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}k^L C)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}k^L.1)^{p_k}]^{\frac{1}{G}}$$

$$;\sum_{j\in K_{1,k}} + \sum_{j\in K_{2,k}} \leq \sum_{j=1}^{k} + \sum_{j=1}^{k}$$

$$\leq [C^G k_i^{L.G}\sum_{k_{i-1}<k\leq k_i}{a_i}^{-1}]^{\frac{1}{G}} + [k_i^{L.G}\sum_{k_{i-1}<k\leq k_i}{a_i}^{-1}]^{\frac{1}{G}} \quad ; k^{L.G} \leq {k_i}^{L.G}$$

$$< [C^G k_i^{L.G}\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^2}]^{\frac{1}{G}} + [k_i^{L.G}\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^2}]^{\frac{1}{G}} \quad ; a_i > k^2 \Rightarrow a_i^{-1} < k^{-2}$$

$$= (C^G k_i^{L.G})^{\frac{1}{G}} (\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}} + (k_i^{L.G})^{\frac{1}{G}} (\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}$$

$$= [(C.k_i^L) + k_i^L] (\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}$$

$$= [k_i^L(C+1)] (\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}.$$

Hence, $[\sum_{k_{i-1}<k\le k_i} (\frac{1}{k}\sum_{j=1}^k \|y_j\|)^{p_k}]^{\frac{1}{G}} \le T_{i,1}(\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}$ then $\sum_{k_{i-1}<k\le k_i} (\frac{1}{k}\sum_{j=1}^k \|y_j\|)^{p_k} \le T_{i,1}^G \sum_{k_{i-1}<k\le k_i} \frac{1}{k^2}$. Therefore, $\sum_{k=1}^{\infty} (\frac{1}{k}\sum_{j=1}^k \|y_j\|)^{p_k} \le T_{i,1}^G \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, where $T_{i,1} = [k_i^L(C+1)]$.

Case $\alpha \ge 1$;

$$[\sum_{k_{i-1}<k\le k_i} (\frac{1}{k}\sum_{j=1}^k \|y_j\|)^{p_k}]^{\frac{1}{G}}$$

$$= [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}} (\sum_{j=1}^k \|a_i^{-1}\alpha^{t_j} j^{t_j} |f_j(x_j)|^{-1}.x_j\|)^{p_k}]^{\frac{1}{G}}$$

$$= [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}} (\sum_{j=1}^k a_i^{-1}\alpha^{t_j} j^{t_j} |f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ; \|x_j\| = 1$$

$$\le [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}} (\sum_{j=1}^k a_i^{-1}\alpha^{\sup_k t_k}. j^{t_j} |f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ; \alpha^{t_j} \le \alpha^{\sup_k t_k}, \alpha \ge 1$$

$$\le [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}} (\sum_{j=1}^k a_i^{-1}\alpha^{\sup_k t_k}. k^{\sup_k t_k} |f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ; j^{t_j} \le k^{t_j} \le k^{\sup_k t_k}.$$

Now doing same as case $\alpha < 1$, which separate by partition. We have

$$= [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}} (\sum_{j\in K_{1,k}} a_i^{-1}\alpha^{\sup_k t_k} k^{\sup_k t_k} |f_j(x_j)|^{-1}$$

$$+ \sum_{j\in K_{2,k}} a_i^{-1}\alpha^{\sup_k t_k} k^{\sup_k t_k} |f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$\le [C^G(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} a_i{}^{-1}]^{\frac{1}{G}} + [(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} a_i{}^{-1}]^{\frac{1}{G}} \quad ; k \le k_i$$

$$< [C^G(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} k^{-2}]^{\frac{1}{G}} + [(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} k^{-2}]^{\frac{1}{G}}$$

$$; a_i > k^2 \Rightarrow a_i^{-1} < k^{-2}$$

$$= [(\alpha.k_i)^L(C+1)] (\sum_{k_{i-1}<k\le k_i} k^{-2})^{\frac{1}{G}}$$

Thus, $[\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k}]^{\frac{1}{G}} \le T_{i,2}(\sum_{k_{i-1}<k\le k_i}\frac{1}{k^2})^{\frac{1}{G}}$ then $\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k} \le T_{i,2}^G\sum_{k_{i-1}<k\le k_i}\frac{1}{k^2}$. Therefore, $\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k} \le T_{i,2}^G\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$, where $T_{i,2} = [(\alpha.k_i)^L(C+1)]$. We obtained $y \in Ces(X,p)$.

For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\le k_i}|f_k(y_k)| &= \sum_{k_{i-1}<k\le k_i}|f_k(a_i^{-1}(\sup_n|f_n(x_n).k|)^{t_k}.|f_k(x_k)|^{-1}x_k)| \\
&= a_i^{-1}\sum_{k_{i-1}<k\le k_i}(\sup_n|f_n(x_n).k|)^{t_k}.b_i^{-t_k}.b_i^{t_k} \quad ;b_i^{-t_k}.b_i^{t_k}=1 \\
&\ge a_i^{-1}\sum_{k_{i-1}<k\le k_i}(\sup_n|f_n(x_n).k|)^{t_k}.b_i^{-t_k} \quad ;b_i^{t_k}\ge 1, b_i > 2^i \\
&= 1.
\end{aligned}$$

Then, $\sum_{k=1}^{\infty}|f_k(y_k)| = \infty$ which is contradiction. The proof is complete. □

Remark 3.2. For each $T_{i,1}$ and $T_{i,2}$ in Proposition 3.1 that are bounded.

$$\begin{aligned}
T_{i,1} &= [k_i^L(C+1)] = [k_i^{\sup_k t_k}(\max_j|f_j(x_j)|^{-1}+1)] \\
T_{i,2} &= [(\alpha.k_i)^L(C+1)] = [(\sup_n|f_n(x_n)|)^{\sup_k t_k}k_i^{\sup_k t_k}(\max_j|f_j(x_j)|^{-1}+1)].
\end{aligned}$$

Proof. Since (f_k) be a sequence of continuous linear functional and in the proof we choose sequence x_k in X with $\|x_k\| = 1$, so $|f_k(x_k)| \le \|f_k\|.\|x_k\| = \|f_k\|.1 < \infty$. We have $|f_k(x_k)|$ bounded, thus $\max_j|f_j(x_j)|^{-1}$ and $(\sup_n|f_n(x_n)|)$ are bounded for all $j = 1,2,...k$. Therefore, $T_{i,1}$ and $T_{i,2}$ are bounded for all $i \in N$. □

4. Main results

In this section, we characterize matrix transformations from $Ces(X,p)$ into Maddox sequence spaces $\ell(q)$ and $\ell_\infty(q)$. By using Lemma 2.1 and Proposition 3.1, we have following theorems.

Theorem 4.1. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to \ell(q)$ if and only if*

(1) *for each $n \in N$ there exists $B_n \in N$ such that*

$$\sum_{k=1}^{\infty}(\sup_j\|f_j^n\|.k)^{t_k}B_n^{-t_k} < \infty,$$

(2) *for each $k \in N$, $\sum_{n=1}^{\infty}|f_k^n(x)|^{q_n} < \infty$ for every $x \in X$ and*

(3) *for each* $r \in N$ *there exists* $B_r \in N$ *such that*

$$\sum_{k\in K}(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|)^{p_k} < \frac{1}{B_r} \quad \Rightarrow \quad \sum_{n=1}^{\infty}|\sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r}$$

for all $x = (x_k) \in \Phi(X)$ *and all finite subsets* K *of* N.

Theorem 4.2. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : Ces(X,p) \to \ell_\infty(q)$ *if and only if there exists* $B \in N$ *such that*

$$\sup_n(\sum_{k=1}^{\infty}(\sup_j \|f_j^n\|.k)^{t_k} B^{-t_k})^{q_n} < \infty.$$

Proof. By Proposition 3.1, we have $\sum_{k=1}^{\infty}(\sup_j \|f_j\|.k)^{t_k} B^{-t_k} < \infty$ for all $x = (x_k) \in Ces(X,p)$. Since $A = (f_k^n) : Ces(X,p) \to \ell_\infty(q)$ and definition of $\ell_\infty(q)$, we have that $\sup_n(\sum_{k=1}^{\infty}(\sup_j \|f_j^n\|.k)^{t_k} B^{-t_k})^{q_n} < \infty$. □

Theorem 4.3. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers such that* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$, *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : Ces(X,p) \to M_\infty(q)$ *if and only if*

(1) *for each* $m, n \in N$ *there exists* $B \in N$ *such that*

$$\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}}(\sup_j \|f_j^n\|)^{t_k}.k^{t_k} B^{-t_k} < \infty$$

(2) *for all* $m, k \in N$, $\sum_{n=1}^{\infty} m^{\frac{1}{q_n}}|f_k^n(x)| < \infty$ *for every* $x \in X$ *and*

(3) *for each* $m, r \in N$ *there exists* $S \in N$ *such that*

$$\sum_{k\in K}(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|)^{p_k} < \frac{1}{S} \quad \Rightarrow \quad \sum_{n=1}^{\infty} m^{\frac{1}{q_n}}|\sum_{k\in K} f_k^n(x_k)| < \frac{1}{r}\ ,$$

for all $x = (x_k) \in \Phi(X)$ *and all finite subsets* K *of* N.

Proof. By [1, Proposition 2.3(vii)], we have $M_\infty(q) = \bigcap_{m=1}^{\infty} \ell_{(m^{\frac{1}{q_n}})}$. By [1, Proposition 2.2(ii) and (iv)] and Theorem 4.1, we have

$$\begin{aligned} A : Ces(X,p) \to M_\infty(q) \quad &\Leftrightarrow \quad A : Ces(X,p) \to \bigcap_{m=1}^{\infty} \ell_{(m^{\frac{1}{q_n}})} \\ &\Leftrightarrow \quad A : Ces(X,p) \to \ell_{(m^{\frac{1}{q_n}})}\ , \text{ for all } m \in N \\ &\Leftrightarrow \quad (m^{\frac{1}{q_n}} f_k^n)_{n,k} : Ces(X,p) \to \ell\ , \text{ for all } m \in N \\ &\Leftrightarrow \quad \text{the conditions (1), (2) and (3) hold.} \end{aligned}$$

□

Theorem 4.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to \underline{\ell_\infty}(q)$ if and only if for each $m \in N$ there exists $B_m \in N$ such that*

$$\sup_n (\sum_{k=1}^{\infty} s^{\frac{t_k}{q_n}} (\sup_j \|f_j^n\|)^{t_k} . k^{t_k} B_m^{-t_k}) < \infty.$$

Proof. By [1, Proposition 2.3(vi)], we have $\underline{\ell_\infty}(q) = \bigcap_{s=1}^{\infty} \ell_{\infty}{}_{(s^{\frac{1}{q_n}})}$ and [1, Proposition 2.2(ii) and (iv)], we have

$$\begin{aligned} A : Ces(X,p) \to \underline{\ell_\infty}(q) &\Leftrightarrow A : Ces(X,p) \to \bigcap_{s=1}^{\infty} \ell_{\infty}{}_{(s^{\frac{1}{q_n}})} \\ &\Leftrightarrow A : Ces(X,p) \to \ell_{\infty}{}_{(s^{\frac{1}{q_n}})}, \text{ for all } s \in N \\ &\Leftrightarrow (s^{\frac{1}{q_n}} f_k^n)_{n,k} : Ces(X,p) \to \ell_\infty, \text{ for all } s \in N \\ &\Leftrightarrow \text{the condition holds.} \end{aligned}$$

□

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