Matrix Transformations on Cesaro Vector-Valued Sequence Space

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Abstract. The purpose of this paper is to find $\beta$-dual of Cesaro vector-valued sequence space and give matrix characterizations from $C_{es}(X,p)$ into sequence spaces $\ell(q)$, $M_\infty(q)$ and $\ell_\infty(q)$ where $p = (p_k)$ is a bounded sequence of positive real numbers such that $p_k > 1$ for all $k \in N$.

1. Introduction

Let $(X,\|\|)$ be a Banach space with a scalar field $K$, the space of all sequences in $X$ is denote by $W(X)$ and let $\Phi(X)$ denote the space of all finite sequences in $X$. When $X = \mathbb{R}$ or $\mathbb{C}$, the corresponding spaces are written as $W$ and $\Phi$. Let $N$ be the set of all natural numbers, we write $x = (x_k)$ with $x_k \in X$ for all $k \in N$. A sequence space in $X$ is linear subspace of $W(X)$. Let $p = (p_k)$ be a bounded sequence of positive real numbers, the $X$-valued sequence space $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $C_{es}(X,p)$, $\ell_\infty(X,p)$, $E_r(X,p)$ and $F_r(X,p)$ are define by:

\begin{align*}
c_0(X,p) &= \{x = (x_k) : \lim_{k \to \infty} \|x_k\|^{p_k} = 0\}, \\
c(X,p) &= \{x = (x_k) : \lim_{k \to \infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}, \\
\ell_\infty(X,p) &= \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\},
\end{align*}

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\[ \ell(X, p) = \{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^p < \infty \}, \]

\[ Ces(X, p) = \{ x = (x_k) : \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{n=1}^{k} \|x_n\|^p \right) < \infty \}, \]

\[ \ell_\infty(X, p) = \{ x = (x_k) : \lim_{k \to \infty} \|\delta_k x_k\| = 0 \text{ for each } (\delta_k) \in c_0 \}, \]

\[ E_r(X, p) = \{ x = (x_k) : \sup_{k} k^{-r} \|x_k\|^p < \infty \}, \text{ and} \]

\[ F_r(X, p) = \{ x = (x_k) : \sum_{k=1}^{\infty} k^r \|x_k\|^p < \infty \}. \]

When \( X = K \), the scalar field of \( X \), the corresponding spaces are written as \( c_0(p), c(p), \ell_\infty(p), Ces(p), \ell_r(p), E_r(p) \) and \( F_r(p) \) respectively.

Grosse and Erdmann [2-3] investigated and gave characterization for infinite matrices to transform between sequence spaces of Maddox. In 1993, F. M. Khan and M. A. Khan [4] gave characterization of infinite matrices of Cesàro sequence space \((Ces(p, s))\) into the space of convergent series \((cs)\) and the space of bounded series \((bs)\). Wu and Liu [9] gave the matrix transformations from \( X \)-valued sequence spaces \( c_0(X, p), \ell_\infty(X, p) \) and \( \ell(X, p) \) into scalar-valued sequence spaces \( c_0(q) \) and \( \ell_\infty(q) \). S. Suantai [6, 7, 8] gave characterization of infinite matrices mapping Nakano vector-valued sequence space \( \ell(X, p) \) into any \( BK \)-space, \( \ell_\infty \) and \( \ell_\infty(q) \). In [1] C. Sudsukh characterized an infinite matrix that transform Maddox vector-valued sequence space into Nakano sequence space and Nakano vector-valued sequence space into Maddox sequence space. In [3] S. Kongnual characterized the matrix transformation of bounded variation vector-valued sequence space into Maddox sequence space.

However, there are many open problems about matrix transformations from vector-valued sequence spaces into scalar-valued sequence spaces. In this paper we study matrix transformations of Cesaro vector-valued sequence space \( Ces(X, p) \) into sequence spaces \( \ell(q), \ell_\infty(q), M_\infty(q) \) and \( \ell_\infty(q) \).

2. Definitions and lemmas

For \( z \in X \) and \( k \in N \), we let \( e^{(k)}(z) \) be the sequence \((0, 0, 0, \ldots, 0, z, 0, \ldots)\) with \( z \) in the \( k^{th} \) position. For a fixed scalar sequence \( u = (u_k) \) the sequence space \( E_u \) is defined by

\[ E_u = \{ x = (x_k) \in W(X) : (u_k x_k) \in E \}. \]

Suppose that the \( X \)-valued sequence space \( E \) is endowed with some linear topology \( \tau \). Then \( E \) is called a \textbf{K-space} if for each \( n \in N \) the \( n^{th} \) coordinate mapping \( p_n : E \to X \), defined by \( p_n(x) = x_n \), is continuous on \( E \). If, in addition, \((E, \tau)\) is a Fréchet(Banach, LF-, LB-) space, then \( E \) is called an \( FK-(BK-, LFK-, LBK-) \) space. Now, suppose that \( E \) contains \( \Phi(X) \). Then \( E \) is said to have \textbf{property AB}
if the set \( \{ \sum_{n=1}^{\infty} e_k(x_n) : n \in N \} \) is bounded in \( E \) for every \( x = (x_k) \in E \). It said to have **property AK** if \( \sum_{k=1}^{\infty} e_k(x_k) \to x \in E \) as \( n \to \infty \) for every \( x = (x_k) \in E \). It has **property AD** if \( \Phi(X) \) is dense in \( E \). Let \( A = (f_k^n) \) with \( f_k^n \) in \( \mathcal{X}' \), the topological dual of \( X \). Suppose that \( E \) is a space of \( X \)-valued sequences and \( F \) a space of scalar-valued sequences. Then \( A \) is said to **map** \( E \) **into** \( F \), written \( A : E \to F \) if for each \( x = (x_k) \in E \), \( A_n(x) = \sum_{k=1}^{\infty} f_k^n (x_k) \) converges for each \( n \in N \) and if the sequence \( Ax = (A_n(x)) \in F \). We denote by \( (E, F) \) the set of all infinite matrices mapping \( E \) into \( F \). If \( u = (u_k) \) and \( v = (v_k) \) are scalar sequences, let
\[
u(E, F)_v = \{ A = (f_k^n) : (u_n v_k f_k^n)_{n, k} \in (E, F) \}.
\]
If \( u_k \neq 0 \) for all \( k \in N \), we write \( u^{-1} = \frac{1}{u_k} \).

In C. Sudsukh [1], this lemma is useful to characterize condition of matrix transformations.

**Lemma 2.1.** Let \( E \subseteq W(X) \) be an FK-space with AK property and \( F \) an FK-space of scalar sequences. Then, for an infinite matrix \( A = (f_k^n), A : E \to F \) if and only if

1. for each \( n \in N, \sum_{k=1}^{\infty} f_k^n (x_k) \) converges for all \( x = (x_k) \in E \),
2. for each \( k \in N, (f_k^n(z))_{n=1}^{\infty} \in F \) for all \( z \in X \), and
3. \( A : \Phi(X) \to F \) is continuous when \( \Phi(X) \) is considered as a subspace of \( E \).

### 3. Some auxiliary results

In this part we first give useful results that concern with \( \beta - \text{dual} \) of \( \text{Ces}(X, p) \).

**Proposition 3.1.** Let \( (f_k) \) be a sequence of continuous linear functional on \( X \) and \( p = (p_k) \) of positive real numbers with \( p_k > 1 \) and \( \frac{1}{p_k} + \frac{1}{1 - p_k} = 1 \) for all \( k \in N \). Then \( \sum_{k=1}^{\infty} f_k(x_k) \) converges for all \( x = (x_k) \in \text{Ces}(X, p) \) if and only if \( \sum_{k=1}^{\infty} (\sup_n \| f_n \|_k t_k B^{-t_k} < \infty \) for some \( B \in N \).

**Proof.** Suppose that \( \sum_{k=1}^{\infty} (\sup_n \| f_n \|_k t_k B^{-t_k} < \infty \) for some \( B \in N \) then for each \( x = (x_k) \in \text{Ces}(X, p) \)
\[
\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \| f_k \|_k B^{-t_k} B^{-t_k} \frac{1}{k} \| x_k \|
\leq \sum_{k=1}^{\infty} (\| f_k \|_k t_k B^{-t_k} + B^{p_k} \frac{1}{k} \| x_k \|^{p_k})
\leq \sum_{k=1}^{\infty} (\sup_k \| f_k \|_k t_k B^{-t_k} + B^{p_k} \frac{1}{k} \| x_k \|^{p_k})
\leq \sum_{k=1}^{\infty} (\sup_n \| f_n \|_k t_k B^{-t_k} + \sum_{k=1}^{\infty} B^{p_k} \frac{1}{k} \| x_k \|^{p_k})
\[
\sum_{k=1}^{\infty} (\sup_{n} \|f_n\|,k)^{t_k} B^{-t_k} + B^{\operatorname{sup} p_k} \sum_{k=1}^{\infty} \left( \frac{1}{k} \|x_k\| \right)^{p_k} \\
\leq \sum_{k=1}^{\infty} (\sup_{n} \|f_n\|,k)^{t_k} B^{-t_k} + B^{\infty} \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{k=1}^{\infty} \|x_k\| \right)^{p_k} \\
< \infty.
\]

Thus \(\sum_{k=1}^{\infty} f_k(x_k)\) converges.

Assume that \(\sum_{k=1}^{\infty} f_k(x_k)\) converges for each \(x = (x_k) \in Ces(X,p)\). Choose scalar sequence \((t_k)\) with \(|t_k| = 1\) such that \(f_k(t_k x_k) = |f_k(x_k)|, \forall k \in N\). Since \((t_k x_k) \in Ces(X,p)\) by assumption we have \(\sum_{n=1}^{\infty} f_k(t_k x_k)\) converges then

\[\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in Ces(X,p).\]  

We want to show that \(\exists B \in N\) such that \(\sum_{k=1}^{\infty} (\sup_{n} \|f_n\|,k)^{t_k} B^{-t_k} < \infty\). On contrary, suppose that

\[\sum_{k=1}^{\infty} (\sup_{n} \|f_n\|,k)^{t_k} B^{-t_k} = \infty, \quad \forall b \in N.\]  

By (3.2) implies that for each \(k_0 \in N\)

\[\sum_{k > k_0}^{\infty} (\sup_{n} \|f_n\|,k)^{t_k} B_{1}^{-t_k} = \infty, \quad \forall b_1 \in N.\]  

From (3.3) we can choose \(b_2 > b_1\) and \(b_2 > 2^2\) and \(k_2 > k_1\) such that

\[\sum_{k_1 < k \leq k_2} (\sup_{n} \|f_n\|,k)^{t_k} B_2^{-t_k} > k^2.\]  

Doing in this way go on, we have sequence \(1 = k_0 < k_1 < k_2 < ...\) and \(b_1 < b_2 < ..., b_i > 2^i\) such that

\[\sum_{k_{i-1} < k \leq k_i} (\sup_{n} \|f_n\|,k)^{t_k} B_i^{-t_k} > k^2.\]  

Choose \(x_k\) in \(X\) with \(|x_k| = 1\) such that \(\sum_{k_{i-1} < k \leq k_i} (\sup_{n} |f_n(x_n)|,k)^{t_k} B_i^{-t_k} > k^2, \forall i \in N\). Let \(a_i = \sum_{k_{i-1} < k \leq k_i} (\sup_{n} |f_n(x_n)|,k)^{t_k} B_i^{-t_k}\) and \(y = (y_k)\), \(y_k = a_i^{-1}(\sup_{n} |f_n(x_n)|,k)^{t_k} |f_k(x_k)|^{-1} x_k\). Then \(y \in Ces(X,p)\). Let \(\alpha = (\sup_{n} |f_n(x_n)|)\) and \(G = \sup_{k} p_k\), we can separate two cases.
Case $\alpha < 1$;

$$
\left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k} \left( \sum_{j=1}^{k} \left\| y_j \right\|^{p_k} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} = \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} \left\| a_j^{-1} \alpha_j^t j^t \left| f_j(x_j) \right|^{-1} \right\|^{p_k} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}}
$$

$$
= \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} \alpha_j^t j^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} \ ; \left\| x_j \right\| = 1
$$

$$
\leq \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} \alpha_j^t j^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} ; \alpha_j^t \leq \alpha < 1, \ t_j > 1
$$

$$
\leq \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} \alpha_j^t j^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} ; j^t \leq k^t \leq \sup_k t_k \forall k \in \mathbb{N}.
$$

Let $K_{1,k}$ and $K_{2,k}$ are partitions of $\{1, 2, ..., k\}$, if $j \in K_{1,k}$ then $|f_j(x_j)| < 1$ and if $j \in K_{2,k}$ then $|f_j(x_j)| \geq 1$ for all $j = 1, 2, ..., k$. So we have

$$
= \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} + \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}}
$$

$$
\leq \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} + \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}}
$$

$$
: |f_j(x_j)|^{-1} \leq \max_j |f_j(x_j)|^{-1} = C, \text{ where } C > 1
$$

$$
\leq \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} + \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}}
$$

$$
: |f_j(x_j)|^{-1} \leq 1
$$

$$
\leq \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}} + \left[ \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}} \right]^{\frac{1}{\alpha}}
$$

$$
: \sum_{j \in K_{1,k}} + \sum_{j \in K_{2,k}} \leq k \sum_{j=1}^{k} \frac{1}{k^{\alpha}} \left( \sum_{j=1}^{k} a_j^{-1} k^t \left| f_j(x_j) \right|^{-1} \right)^{\frac{1}{p_k}}
$$

$$
\leq \left[ C^{G_k} k^{lG_k} \sum_{k_{i-1} < k \leq k_i} a_i^{-1} \right]^{\frac{1}{\alpha}} + \left[ k^{L} \sum_{k_{i-1} < k \leq k_i} a_i^{-1} \right]^{\frac{1}{\alpha}} ; k^{lG} \leq k^{L} G
$$

$$
< \left[ C^{G_k} k^{lG_k} \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{2}} \right]^{\frac{1}{\alpha}} + \left[ k^{L} \sum_{k_{i-1} < k \leq k_i} \frac{1}{k^{2}} \right]^{\frac{1}{\alpha}} ; a_i > k^2 \Rightarrow a_i^{-1} < k^{-2}
$$
\[
C_k = \left( \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^2} \right)^{\frac{1}{p}} + \left( \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^2} \right)^{\frac{1}{p}} = \left[ (C,k^G) + k^G \right] \left( \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^2} \right)^{\frac{1}{p}} = \left[ k^G(C + 1) \right] \left( \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^2} \right)^{\frac{1}{p}}.
\]

Hence, \[\sum_{k_i - 1 < k_i \leq k_i} \left( \frac{1}{k} \sum_{j=1}^k \|y_j\|_{p_k} \right)^{\frac{1}{p}} \leq T_{i,1} \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{1/p}} \] then \[\sum_{k_i - 1 < k_i \leq k_i} \left( \frac{1}{k^p} \sum_{j=1}^k \|y_j\|_{p_k} \right)^{1/p} \leq T_{i,1} \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{1/p}} \]. Therefore, \[\sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{j=1}^k \|y_j\|_{p_k} \right)^{1/p} \leq T_{i,1} \sum_{k=1}^{\infty} \frac{1}{k^{1/p}} < \infty,\] where \(T_{i,1} = [k^G(C + 1)]\).

Case \(\alpha \geq 1\):

\[
\left[ \sum_{k_i - 1 < k_i \leq k_i} \left( \frac{1}{k} \sum_{j=1}^k \|y_j\|_{p_k} \right)^{\frac{1}{p}} \right]^{\frac{1}{p^*}} = \left[ \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{p^*}} \left( \sum_{j=1}^k \|a_i^{-1} \alpha_j \|_{\infty} \right)^{1/p^*} \right]^{\frac{1}{p^*}} = \left[ \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{p^*}} \left( \sum_{j=1}^k \|a_i^{-1} \alpha_j \|_{\infty} \right)^{1/p^*} \right]^{\frac{1}{p^*}} ; \|x_j\| = 1
\]

\[
\leq \left[ \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{p^*}} \left( \sum_{j=1}^k a_i^{-1} \alpha_j \|x_j\|_{\infty} \right)^{1/p^*} \right]^{\frac{1}{p^*}} ; \alpha_j \leq \alpha_{\text{sup}_k t_k}, \alpha \geq 1
\]

\[
\leq \left[ \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{p^*}} \left( \sum_{j=1}^k a_i^{-1} \alpha_t \|x_j\|_{\infty} \right)^{1/p^*} \right]^{\frac{1}{p^*}} ; j = k_t \leq \left( \sum_{k_i - 1 < k_i \leq k_i} \frac{1}{k^{p^*}} \right)^{\frac{1}{p^*}} .
\]

Now doing same as case \(\alpha < 1\), which separate by partition. We have

\[
\left[ C^G (\alpha, k_i)^{L.G} \sum_{k_i - 1 < k_i \leq k_i} a_i^{-1} \right]^{\frac{1}{p^*}} + \left[ \sum_{k_i - 1 < k_i \leq k_i} a_i^{-1} \right]^{\frac{1}{p^*}} ; k \leq k_i \leq k^2 \Rightarrow a_i^{-1} < k^{-2}
\]

\[
\left[ C^G (\alpha, k_i)^{L.G} \sum_{k_i - 1 < k_i \leq k_i} a_i^{-1} \right]^{\frac{1}{p^*}} + \left[ \sum_{k_i - 1 < k_i \leq k_i} a_i^{-1} \right]^{\frac{1}{p^*}} ; k_i > k^2 \Rightarrow a_i^{-1} < k^{-2}
\]

\[
\left[ (\alpha, k_i)^{L}(C + 1)\right] \left( \sum_{k_i - 1 < k_i \leq k_i} k^{-2} \right)^{\frac{1}{p^*}}
\]
Thus, \[ \sum_{k_i-1 < k < k_i} \left( \frac{1}{p} \sum_{j=1}^{k} \| y_j \|^{p_k} \right)^{\frac{1}{p_k}} \leq T_{i,2} \left( \sum_{k_i-1 < k < k_i} \frac{1}{p_k} \right) \leq T_{i,2} \sum_{k_i-1 < k < k_i} \frac{1}{p_k}. \] Therefore, \[ \sum_{k=1}^{\infty} \left( \frac{1}{p} \sum_{j=1}^{k} \| y_j \|^{p_k} \right)^{\frac{1}{p_k}} \leq T_{i,2} \sum_{k=1}^{\infty} \frac{1}{p_k} < \infty, \] where \( T_{i,2} = [\alpha.k_i]^L(C + 1) \). We obtained \( y \in Ces(X, p) \).

For each \( i \in N \), we have
\[
\sum_{k_i-1 < k < k_i} |f_k(y_k)| = \sum_{k_i-1 < k < k_i} |f_k(a_i^{-1}(\sup_n |f_n(x_n)\cdot k|)^{t_k} \cdot |f_k(x_k)|^{-1} x_k)|
\]
\[
= a_i^{-1} \sum_{k_i-1 < k < k_i} (\sup_n |f_n(x_n)\cdot k|)^{t_k} \cdot b_i^{-t_k} \cdot b_i^{t_k} = 1
\]
\[
\geq a_i^{-1} \sum_{k_i-1 < k < k_i} (\sup_n |f_n(x_n)\cdot k|)^{t_k} \cdot b_i^{-t_k} ; \quad b_i^{t_k} \geq 1, b_i > 2^i
\]
\[
= 1.
\]
Then, \( \sum_{k=1}^{\infty} |f_k(y_k)| = \infty \) which is contradiction. The proof is complete. \( \square \)

**Remark 3.2.** For each \( T_{i,1} \) and \( T_{i,2} \) in Proposition 3.1 that are bounded.

\[
T_{i,1} = [\alpha.k_i]^L(C + 1) = [\alpha.k_i^{sup_p} \cdot t_k \cdot (max_j |f_j(x_j)|^{-1} + 1)]
\]
\[
T_{i,2} = [\alpha.k_i]^L(C + 1) = [\alpha.k_i^{sup_p} \cdot t_k \cdot (sup_n |f_n(x_n)|)^{sup_p} \cdot k_i^{sup_p} \cdot t_k \cdot (max_j |f_j(x_j)|^{-1} + 1)]
\]

**Proof.** Since \( (f_k) \) be a sequence of continuous linear functional and in the proof we choose sequence \( x_k \) in \( X \) with \( \| x_k \| = 1 \), so \( |f_k(x_k)| \leq \| f_k \| \cdot \| x_k \| = \| f_k \| \cdot 1 < \infty \). We have \( |f_k(x_k)| \) bounded, thus \( max_j |f_j(x_j)|^{-1} \) and \( (sup_n |f_n(x_n)|) \) are bounded for all \( j = 1, 2, ... k \). Therefore, \( T_{i,1} \) and \( T_{i,2} \) are bounded for all \( i \in N \). \( \square \)

4. Main results

In this section, we characterize matrix transformations from \( Ces(X, p) \) into Maddox sequence spaces \( \ell(q) \) and \( \ell_\infty(q) \). By using Lemma 2.1 and Proposition 3.1, we have the following theorems.

**Theorem 4.1.** Let \( p = (p_k) \) and \( q = (q_k) \) be bounded sequences of positive real numbers such that \( p_k > 1 \) and \( \frac{1}{p_k} + \frac{1}{q_k} = 1 \) for all \( k \in N \), and \( A = (a_{i,j}^n) \) an infinite matrix. Then \( A : Ces(X, p) \rightarrow \ell(q) \) if and only if

1. \( \) for each \( n \in N \) there exists \( B_n \in N \) such that \( \sum_{k=1}^{\infty} (\sup_j |f_j^n| \cdot k)^{t_k} B_n^{-t_k} < \infty \),

2. \( \) for each \( k \in N, \sum_{n=1}^{\infty} |f_k^n(x)|^{q_n} < \infty \) for every \( x \in X \) and
(3) for each \( r \in N \) there exists \( B_r \in N \) such that
\[
\sum_{k \in K} \left( \frac{1}{k} \sum_{j=1}^{k} \|x_j\|^{p_k} \right) < \frac{1}{B_r} \Rightarrow \sum_{n=1}^{\infty} \left( \sum_{k \in K} f^n_k(x_k) \right)^{q_n} < \frac{1}{r}
\]
for all \( x = (x_k) \in \Phi(X) \) and all finite subsets \( K \) of \( N \).

**Theorem 4.2.** Let \( p = (p_k) \) and \( q = (q_k) \) be bounded sequences of positive real numbers with \( p_k > 1 \) and \( \frac{1}{p_k} + \frac{1}{q_k} = 1 \) for all \( k \in N \) and \( A = (f^n_k) \) an infinite matrix. Then \( A : Ces(X, p) \to \ell_\infty(q) \) if and only if there exists \( B \in N \) such that
\[
\sup_n \left( \sup_j \|f^n_j\| k^{\frac{1}{q_n}} B^{-1/k} \right) < \infty.
\]

**Proof.** By Proposition 3.1, we have \( \sum_{k=1}^{\infty} (\sup_j \|f^n_j\| k^{1/k} B^{-1/k}) < \infty \) for all \( x = (x_k) \in Ces(X, p) \). Since \( A = (f^n_k) : Ces(X, p) \to \ell_\infty(q) \) and definition of \( \ell_\infty(q) \), we have that \( \sup_n (\sum_{k=1}^{\infty} (\sup_j \|f^n_j\| k^{1/k} B^{-1/k})) < \infty. \)

**Theorem 4.3.** Let \( p = (p_k) \) and \( q = (q_k) \) be bounded sequences of positive real numbers such that \( p_k > 1 \) and \( \frac{1}{p_k} + \frac{1}{q_k} = 1 \) for all \( k \in N \), and \( A = (f^n_k) \) an infinite matrix. Then \( A : Ces(X, p) \to M_\infty(q) \) if and only if

1. for each \( m, n \in N \) there exists \( B \in N \) such that
\[
\sum_{k=1}^{\infty} m^{\frac{1}{m_n}} (\sup_j \|f^n_j\| k^{1/k} B^{-1/k}) < \infty
\]
2. for all \( m, k \in N \), \( \sum_{n=1}^{\infty} m^{\frac{1}{m_n}} |f^n_k(x)| < \infty \) for every \( x \in X \)
3. for each \( m, r \in N \) there exists \( S \in N \) such that
\[
\sum_{k \in K} \left( \frac{1}{k} \sum_{j=1}^{k} \|x_j\|^{p_k} \right) < \frac{1}{S} \Rightarrow \sum_{n=1}^{\infty} m^{\frac{1}{m_n}} |f^n_k(x_k)| < \frac{1}{r}
\]
for all \( x = (x_k) \in \Phi(X) \) and all finite subsets \( K \) of \( N \).

**Proof.** By [1, Proposition 2.3(vii)], we have \( M_\infty(q) = \bigcap_{m=1}^{\infty} \ell_{(m \frac{1}{m_n})} \). By [1, Proposition 2.2(ii) and (iv)] and Theorem 4.1, we have

\[
A : Ces(X, p) \to M_\infty(q) \iff A : Ces(X, p) \to \bigcap_{m=1}^{\infty} \ell_{(m \frac{1}{m_n})}
\]
\[
A : Ces(X, p) \to \ell_{(m \frac{1}{m_n})}, \text{ for all } m \in N
\]
\[
(m \frac{1}{m_n} f^n_k)_{n,k} : Ces(X, p) \to \ell, \text{ for all } m \in N
\]
\[
\text{the conditions (1), (2) and (3) hold.}
Theorem 4.4. Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$, and $A = (a^n_k)$ an infinite matrix. Then $A : Ces(X, p) \rightarrow \ell_\infty(q)$ if and only if for each $m \in N$ there exists $B_m \in N$ such that
\[
\sup_n \left( \sum_{k=1}^{\infty} \frac{1}{s^{\frac{1}{p_k}}} (\sup_j \| f_j^n \| s^{\frac{1}{q_k}} k^{j_k} B_m^{-j_k}) \right) < \infty.
\]

Proof. By [1, Proposition 2.3(vi)], we have $\ell_\infty(q) = \bigcap_{n=1}^{\infty} \ell_\infty\left( s^{\frac{1}{p_k}} \right)$ and [1, Proposition 2.2(ii) and (iv)], we have
\[
A : Ces(X, p) \rightarrow \ell_\infty(q) \iff A : Ces(X, p) \rightarrow \bigcap_{n=1}^{\infty} \ell_\infty\left( s^{\frac{1}{q_k}} \right), \text{ for all } s \in N
\]
\[
\iff (s^{\frac{1}{q_k}} f_j^n)_n, k : Ces(X, p) \rightarrow \ell_\infty, \text{ for all } s \in N
\]
\[
\iff \text{the condition holds.}
\]

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References

