Matrix Transformations from Cesaro Vector-Valued Sequence Space into Orlicz Sequence Space
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Abstract. The purpose of this paper, we give matrix characterizations from Cesaro vector-valued sequence space $Ces(X,p)$ into Orlicz sequence space ℓ_M and by using this result we obtain matrix characterizations from $Ces(X,p)$ into h_M and ℓ_r, where $p=(p_k)$ is a bounded sequence of positive real numbers such that $p_k>1$ for all $k\in N$.

1. Introduction

Let $(X,\|.\|)$ be a Banach space with a scalar field K, the space of all sequences in X is denote by $W(X)$ and let $\Phi(X)$ denote the space of all finite sequences in X. When $X=\mathbb{R}$ or $\mathcal{C}$, the corresponding spaces are written as W and Φ. Let N be the set of all natural numbers, we write $x=(x_k)$ with $x_k\in X$ for all $k\in N$. A sequence space in X is linear subspace of $W(X)$. Let $p=(p_k)$ be a bounded sequence of positive real numbers, the X-valued sequence space $c_0(X,p), c(X,p), \ell_\infty(X,p), \ell(X,p), Ces(X,p), \underline{\ell_\infty}(X,p), E_r(X,p)$ and $F_r(X,p)$ are define by:

$$\begin{aligned}
c_0(X,p) &= \{x=(x_k) : \lim_{k\to\infty}\|x_k\|^{p_k}=0\},\\
c(X,p) &= \{x=(x_k) : \lim_{k\to\infty}\|x_k-a\|^{p_k}=0 \text{ for some } a\in X\},\\
\ell_\infty(X,p) &= \{x=(x_k) : \sup_k\|x_k\|^{p_k}<\infty\},\\
\ell(X,p) &= \{x=(x_k) : \sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\},\\
Ces(X,p) &= \{x=(x_k) : \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{k}\|x_n\|)^{p_k}<\infty\},
\end{aligned}$$

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\begin{aligned}
\underline{\ell_\infty}(X,p) &= \{x=(x_k): \lim_{k\to\infty}\|\delta_k x_k\| = 0 \quad \text{for each}(\delta_k)\in c_0\},\\
E_r(X,p) &= \{x=(x_k): \sup_k k^{-r}\|x_k\|^{p_k} < \infty\}, \ \text{ and}\\
F_r(X,p) &= \{x=(x_k): \sum_{k=1}^{\infty} k^r\|x_k\|^{p_k} < \infty\}.
\end{aligned}
$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p), Ces(p), \underline{\ell_\infty}(p), E_r(p)$ and $F_r(p)$ respectively.

Grosse and Erdmann [2, 3] investigated and gave characterization for infinite matrices to transform between sequence spaces of Maddox. In 1993, F. M. Khan and M. A. Khan[4] gave characterization of infinite matrices of Cesàro sequence space $(Ces(p,s))$ into the space of convergent series (cs) and the space of bounded series (bs). S. Suantai[6, 7] gave characterization of infinite matrices mapping Nakano vector-valued sequence space $\ell(X,p)$ into any BK-space, ℓ_∞ and $\ell_\infty(q)$. S. Suantai[9, 11] has given matrix characterizations from $\ell_\infty(X,p)$, $c_0(X)$ and $\ell(X,p)$ into the Orlicz sequence space.

In this paper, we interested to find characterizations from $Ces(X,p)$ into Orlicz sequence space.

2. Notation and Definitions

For $z \in X$ and $k \in N$, we let $e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined by

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a **K-space** if for each $n \in N$ the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E,τ) is a Fréchet(Banach, LF-, LB-) space, then E is called an $FK-(BK-, LFK-, LBK-)$ space. Now, suppose that E contains $\Phi(X)$. Then E is said to have **property AB** if the set $\{\Sigma_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It said to have **property AK** if $\Sigma_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x = (x_k) \in E$. It has **property AD** if $\Phi(X)$ is dense in E. Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to **map** E **into** F, written $A : E \to F$ if for each $x = (x_k) \in E, A_n(x) = \Sigma_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and if the sequence $Ax = (A_n(x)) \in F$. We denote by (E,F)

the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u_k})$.

A function $M : \mathbb{R} \to [0,\infty)$ is said to be an *Orlicz function* if it is even, convex, continuous and vanishing only at 0. We define the Orlicz sequence space by the formula

$$\ell_M = \{x = (x_k) \in \ell^0 : \rho_M(cx) = \sum_{k=1}^{\infty} M(cx_k) < \infty \ for\ some\ c > 0\}$$

where ℓ^0 stands for the space of all real sequences. We consider ℓ_M equipped with the Luxemburg norm

$$\|x\| = \inf\{\varepsilon > 0 : \rho_M(\frac{x}{\varepsilon}) \leq 1\}.$$

Let h_M denote the subspace order continuous elements, i.e

$$h_M = \{x = (x_k) : \rho_M(cx) < \infty \quad for\ any \quad c > 0\}.$$

It is know that ℓ_M is a BK-space and h_M is a closed subspace of ℓ_M.

We say that an Orlicz function M $satisfiesthe$ δ_2 *condition* ($M \in \delta_2$ for short) if there exist constants $k \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

In C. Sudsukh [8], this Proposition is useful to characterize condition of matrix transformations.

Proposition 2.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence space, and let u and v be sequences of real numbers with $u_k \neq 0$ for all $k \in N$. Then we have*

(i) $(\bigcup_{n=1}^{\infty} E_n, F) = \bigcap_{n=1}^{\infty}(E_n, F)$,

(ii) $(E, \bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty}(E, F_n)$,

(iii) $(E_u, F_v) =_v (E, F,)_{u^{-1}}$,

(iv) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$,

(v) $(E, F_1) = (E, F_2) \cap (\Phi(X), F_1)$ *if* E *an* FK*-space with* AD, F_2 *is an* FK*-space and* F_1 *is a closed subspace of* F_2.

Proposition 2.2. *Let* M *be Orlicz function and* $x \in \ell_M$.

(i) *If* $\|x\| \leq 1$, *then* $\rho_M(x) \leq \|x\|$.

(ii) *If* $\|x\| > 1$, *then* $\rho_M(x) > \|x\|$.

(iii) *If* $M \in \delta_2$, *then* $\|x\| = 1 \Rightarrow \rho_M(x) = 1$.

Proof. See [1, Theorem 1.38 and Theorem 1.39] □

3. Main Results

We first give characterizations of infinite matrices mapping the Cesaro vector-valued sequence space $Ces(X, p)$ into the Orlicz sequence space.

Theorem 3.1. *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers with* $p_k > 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A \in (Ces(X, p), \ell_M)$ *if and only if*

(1) *for each* $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$ *for all* $x \in X$ *and*

(2) *there exists* $m_0 \in N$ *such that*

$$\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} (M \circ (m_0^{-p_k} f_k^n))(x) \leq 1.$$

Proof. Assume that $A \in (Ces(X, p), \ell_M)$, we want to show conditions hold. Since $e^k(x) \in Ces(X, p)$ for all $x \in X$ and all $k \in N$, we have $Ae^k(x) \in \ell_M$, so (1) is obtained. By Zeller's theorem, we have that $A : Ces(X, p) \to \ell_M$ is continuous. Then there exists $m_0 \in N$ such that

$$x = (x_k) \in Ces(X, p), \quad \|x\| \leq \tfrac{1}{m_0} \;\Rightarrow\; \|Ax\| \leq 1. \tag{3.1}$$

Let $x \in X$ with $\|x\| \leq 1$ and $k \in N$. We have $m_0^{-p_k} e^k(x) \in Ces(X, p)$ and $\|m_0^{-p_k} e^k(x)\| \leq \frac{1}{m_0}$. By (3.1) we have $\|(m_0^{-p_k} f_k^n(x))_{n=1}^{\infty}\| \leq \|A(m_0^{-p_k} e^k(x))\| \leq 1$. By Proposition 2.2(i) we obtain that $\sum_{n=1}^{\infty} M(m_0^{-p_k} f_k^n(x)) \leq 1$. This implies that

$$\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} (M \circ (m_0^{-p_k} f_k^n))(x) \leq 1\ ,$$

thus condition (2) holds.

Conversely, assume that the conditions (1) and (2) hold. By (2), there exists $m_0 \in N$ such that $\sum_{n=1}^{\infty} M(m_0^{-p_k} f_k^n(x)) \le 1$ for all $k \in N$ and all $x \in X$ with $\|x\| \le 1$. From Proposition 2.2(i) we have that $\|A(m_0^{-p_k} e^k(x))\| \le \|(m_0^{-p_k} f_k^n(x))_{n=1}^{\infty}\| \le 1$ for all $k \in N$ and all $x \in X$ with $\|x\| \le 1$. Hence, for $x \in X$ with $x \neq 0$, we have

$$\|A(m_0^{-p_k} e^k(x))\| \ \le \ \|(m_0^{-p_k} f_k^n(x))_{n=1}^{\infty}\| \ \le \ \|x\| \tag{3.2}$$

Let $x = (x_k) \in Ces(X, p)$ and $k \in N$. By (3.2) we have

$$\begin{aligned} \|Ae^k(x_k)\| &= \|A(m_0^{-p_k} \cdot m_0^{p_k} e^k(x_k))\| \\ &= m_0^{p_k} \|A(m_0^{-p_k} \cdot e^k(x_k))\| \\ &\le m_0^{p_k} \cdot \|x_k\| \\ &\le m_0^{p_k} \cdot \sum_{n=1}^{k} \|x_n\| \end{aligned} \tag{3.3}$$

Since, for all $x = (x_k) \in Ces(X, p)$ we have that $\sum_{k=1}^{\infty} (\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k} < \infty$ when $p_k > 1$. Let $\sum_{k=1}^{\infty} (\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k} = L$. Fixed for each k, so

$$(\sum_{n=1}^{k} \|x_n\|) \le L. \tag{3.4}$$

By (3.3) and (3.4) we have,

$$\begin{aligned} \sum_{k=1}^{\infty} \|Ae^k(x_k)\| &\le \sum_{k=1}^{\infty} m_0^{p_k} \cdot (\sum_{n=1}^{k} \|x_n\|) \\ &= \sum_{k=1}^{\infty} m_0^{p_k} \cdot \frac{(\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)}{\frac{1}{k}} \cdot \frac{(\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k}}{(\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k}} \\ &= \sum_{k=1}^{\infty} m_0^{p_k} \cdot (\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k} \cdot \frac{1}{\frac{1}{k}(\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k - 1}} \\ &\le m_0^{\sup_k p_k} \sum_{k=1}^{\infty} (\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k} \cdot \frac{1}{((\frac{1}{k})^{\frac{p_k}{p_k - 1}} \sum_{n=1}^{k} \|x_n\|)^{p_k - 1}} \end{aligned}$$

$$
\begin{aligned}
&= m_0^G \sum_{k=1}^{\infty} (\frac{1}{k}\sum_{n=1}^{k} \|x_n\|)^{p_k} \cdot ((\frac{1}{k})^{\frac{p_k}{p_k-1}} \sum_{n=1}^{k} \|x_n\|)^{1-p_k} \ ; G = \sup_k p_k \\
&\le m_0^G \sum_{k=1}^{\infty} (\frac{1}{k}\sum_{n=1}^{k} \|x_n\|)^{p_k} (1 \cdot \sum_{n=1}^{k} \|x_n\|)^{1-p_k} \\
&\le m_0^G \sum_{k=1}^{\infty} (\frac{1}{k}\sum_{n=1}^{k} \|x_n\|)^{p_k} (L)^{1-p_k} \qquad ; \ L = \max(1, \inf_k L) \\
&\le m_0^G \cdot L^{1-\inf_k p_k} \sum_{k=1}^{\infty} (\frac{1}{k}\sum_{n=1}^{k} \|x_n\|)^{p_k} \\
&= V \sum_{k=1}^{\infty} (\frac{1}{k}\sum_{n=1}^{k} \|x_n\|)^{p_k} \qquad ; V = m_0^G L^{1-M}, \ M = \inf_k p_k \\
&< \infty.
\end{aligned}
$$

Therefore $\sum_{k=1}^{\infty} Ae^k(x_k)$ converges absolutely in ℓ_M. Since ℓ_M is Banach, $\sum_{k=1}^{\infty} Ae^k(x_k)$ converges in ℓ_M. Let $y = (y_k) \in \ell_M$ be the sum of the series $\sum_{k=1}^{\infty} Ae^k(x_k)$. By continuity of p_m, we have for each $m \in N$,

$$
y_m = p_m(y) = \lim_{n\to\infty} \sum_{k=1}^{n} p_m(Ae^k(x_k)) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k^m(x_k)
$$

This implies that Ax exists and $(Ax)_m = \sum_{k=1}^{\infty} f_k^m(x_k) = y_m$, so that $Ax \in \ell_M$.

Theorem 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A \in (Ces(X,p), h_M)$ if and only if*

(1) *for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in h_M$ for all $x \in X$ and*

(2) *there exists $m_0 \in N$ such that*

$$
\sup_k \sup_{\|x\|\le 1} \sum_{n=1}^{\infty} (M \circ (m_0^{-p_k} f_k^n))(x) \le 1.
$$

Proof. Since h_M is a closed subspace of ℓ_M, the theorem is obtained by applying Theorem 3.1 and Proposition 2.1(v). □

Theorem 3.3. *For an infinite matrix $A = (f_k^n)$, $A \in (Ces(X), \ell_M)$ if and only if*

(1) *for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$ for all $x \in X$ and*

(2) *there exists $m_0 \in N$ such that*

$$\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} (M \circ (m_0^{-1} f_k^n))(x) \leq 1.$$

By Theorem 3.1 and $M(t) = |t|^r$, $r \geq 1$, we have $\ell_M = \ell_r$. We have this result.

Corollary 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$ and $r \geq 1$. Then for an infinite matrix $A = (f_k^n)$, $A \in (Ces(X, p), \ell_r)$ if and only if*

(1) *for each $k \in N$, $\sum_{n=1}^{\infty} |f_k^n(x)|^r < \infty$ for all $x \in X$ and*

(2) *there exists $m_0 \in N$ such that*

$$\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |m_0^{-p_k} f_k^n(x)|^r \leq 1.$$

Corollary 3.5. *For $r \geq 1$ and for an infinite matrix $A = (f_k^n)$, $A \in (Ces(X), \ell_r)$ if and only if*

(1) *for each $k \in N$, $\sum_{n=1}^{\infty} |f_k^n(x)|^r < \infty$ for all $x \in X$ and*

(2) $\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |f_k^n(x)|^r < \infty$.

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